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On the stochastic p-Laplace equation ☆

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ABSTRACT

The p-Laplace equation with random perturbation is studied for the singular case $1 < p \leqslant 2$ in this paper. Some properties of the invariant measure and transition semigroups are obtained. The main tool is the dimension-free Harnack inequality, which is established by using the coupling argument. As consequences, some ergodicity, compactness and contractive properties are derived for the associated transition semigroups. The main results are applied to stochastic reaction–diffusion equations and the stochastic p-Laplace equation in Hilbert space.

1. Introduction and main results

In this paper we study the stochastic p-Laplace equation with some nonlinear perturbation in the drift. In the field of nonlinear PDE, the following p-Laplace equation

$$\partial_t u = \mathbf{div}\left(|\nabla u|^{p-2}\nabla u\right), \quad 1 < p < \infty \tag{1.1}$$

has been studied intensively, we mainly refer to [9,33] and references therein. (1.1) describes the type of diffusion with diffusivity depending on the gradient of the main unknown, and it also has a strong connection with the porous media and fast diffusion equations (cf. [9,33,34]). This type of equation arises from geometry, quasiregular mappings and fluid dynamics etc. (cf. [9]). In particular, Ladyzenskaja suggests (1.1) as a model of motion of non-newtonian fluids in [18]. In the stochastic case, the existence and uniqueness of solution for stochastic p-Laplace equation follows from the general results in [15,28,40]. The large deviation principle has been established in [20] for (1.1) with small multiplicative noise. For the degenerate case (i.e. $p > 2$), the Markov property of solution and invariant measure have been studied in [27].

In this work we will adopt the variational approach to analyze the stochastic p-Laplace equation. Comparing with the martingale measure approach [35] and the semigroup approach [8], one advantage of the variational approach is it can give a unified framework to treat a wide class of quasi-linear SPDEs, such as stochastic reaction–diffusion equations, stochastic porous media equations and the stochastic p-Laplace equation. This approach was initiated by Bensoussan and Temam [5] in additive noise case, then it was further developed by Pardoux for linear SPDE in [26] and Krylov and Rozovskii for general stochastic evolution equations (SEE) in [15]. This classical framework was extended later in different directions: e.g. in [11] for SEE driven by general martingale, in [28] for SEE with coefficients related to Orlicz space framework, and in [40] for SEE

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with random coercivity constant. The Galerkin approximation and monotonicity method are the main ingredients in those works.

The main aim of this paper is to establish the ergodicity, ultrabounded and compact property for the associate transition semigroups. In particular, this implies that the corresponding Kolmogorov operator of the stochastic p-Laplace equation in Hilbert space only has discrete spectrum. Moreover, we also derive the uniqueness of invariant measures and the convergence rate of the semigroup to its equilibrium. In particular, this implies the decay estimate of the solution to the deterministic p-Laplace equation with absorption term.

The main tool is the dimension-free Harnack inequality, which was first introduced by Wang in [36] for diffusion semigroups on manifolds with curvature bounded from below. Later, this type of Harnack inequality turns out to be a very efficient tool in the study of finite and infinite-dimensional diffusion semigroups. We refer to [1,29,30,36] for applications to contractivity properties and functional inequalities; [2,3,14] for applications to short-time asymptotics of infinite-dimensional diffusions: [4,12,19,39] for applications to heat kernel estimates and [6] for the study of transportation cost inequalities.

Recently, the dimension-free Harnack inequality was established in [39] for stochastic porous media equations and in [19] for stochastic fast diffusion equations. As applications, the strong Feller property and some contractive properties were obtained for the corresponding transition semigroups. In [19,39] the authors employed an approach consists of coupling method and Girsanov transformation, which was first developed in [4] for diffusion semigroups on Riemannian manifolds with unbounded below curvatures. This coupling argument can avoid the curvature bounds condition and the gradient estimate, which were essentially used in previous works (cf. [2,3,6,29,30]) and could be very hard to verify in the framework of SPDE.

Let Λ be an open bounded domain in $\mathbb{R}^d$ with a C^1 boundary. Consider the following Gelfand triple

$$H_0^{1,p}(\Lambda)\cap L^q(\Lambda)\subseteq L^2(\Lambda)\subseteq \big(H_0^{1,p}(\Lambda)\cap L^q(\Lambda)\big)^*$$

and the stochastic p-Laplace equation

$$\mathrm{d}X_t=\big[\mathbf{div}\big(|\nabla X_t|^{p-2}\nabla X_t\big)-\gamma|X_t|^{q-2}X_t\big]\,\mathrm{d}t+B\,\mathrm{d}W_t,\quad X_0=x, \tag{1.2}$$

where $1<p\leqslant 2\leqslant q$ and $\gamma\geqslant 0$, B is a Hilbert–Schmidt operator on $L^2(\Lambda)$ and W_t is a cylindrical Wiener process on $L^2(\Lambda)$ w.r.t. a complete filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. One should note that we would remove $L^q(\Lambda)$ in the Gelfand triple if $\gamma=0$ in (1.2).

Since Λ is a bounded domain, by the Poincaré inequality we know that the following norm

$$\|u\|_{1,p}:=\bigg(\int_\Lambda \big|\nabla u(\xi)\big|^p\,\mathrm{d}\xi\bigg)^{1/p},\quad u\in H_0^{1,p}(\Lambda)$$

is equivalent to the classical Sobolev norm in $H_0^{1,p}(\Lambda)$. For simplicity, we will use this equivalent norm in this paper. We denote the norm in $L^r(\Lambda)$ by $\|\cdot\|_r$ and the inner product in $L^2(\Lambda)$ by $\langle\cdot,\cdot\rangle$.

According to [40, Theorem 3.6], it is easy to show the coefficients of (1.2) satisfy the so-called monotone and coercive conditions (see also [15,27]). Hence for any $x\in L^2(\Lambda)$ Eq. (1.2) has a unique solution $X_t(x)$, which is a continuous adapted process on $L^2(\Lambda)$ and satisfies

$$\mathbb{E}\bigg(\sup_{t\in[0,T]}\big\|X_t(x)\big\|_2^2+\int_0^T\big\|X_t(x)\big\|_{1,p}^p\,\mathrm{d}t\bigg)<\infty,\quad T>0. \tag{1.3}$$

Moreover, we have the crucial Itô formula

$$\|X_t\|_2^2=\|X_0\|_2^2+\int_0^t\big(b-2\|X_s\|_{1,p}^p-2\gamma\|X_s\|_q^q\big)\,\mathrm{d}s+2\int_0^t\langle X_s,B\,\mathrm{d}W_s\rangle,\quad t\geqslant 0, \tag{1.4}$$

where $b=\|B\|_{\mathrm{HS}}^2$ (Hilbert–Schmidt norm).

Now we consider the associated transition semigroups

$$P_tF(x):=\mathbb{E}F\big(X_t(x)\big),\qquad F\in\mathfrak{M}_b\big(L^2(\Lambda)\big),\quad t>0,$$

where $\mathfrak{M}_b(L^2(\Lambda))$ is the class of all bounded measurable functions on $L^2(\Lambda)$.

Theorem 1.1. *Suppose the embedding $H_0^{1,p}(\Lambda)\subseteq L^2(\Lambda)$ is compact.*

(i) *The transition semigroup $\{P_t\}$ has an invariant probability measure.*

(ii) *If $\gamma > 0$, then $\{P_t\}$ has a unique invariant measure μ. Moreover, for some $\varepsilon_0 > 0$ we have*

$$\mu\big(\|\cdot\|_{1,p}^p + e^{\varepsilon_0\|\cdot\|_2^q}\big) < \infty.$$

(iii) *If $\gamma > 0$ and $q = 2$, then for any Lipschitz continuous function F on $L^2(\Lambda)$ we have*

$$\big|P_t F(x) - \mu(F)\big| \leqslant \mathrm{Lip}(F) e^{-\gamma t}\big(\|x\|_2 + C\big), \quad t \geqslant 0,\ x \in L^2(\Lambda), \tag{1.5}$$

where C is a constant and $\mathrm{Lip}(F)$ *is the Lipschitz constant of F.*

(iv) *If $\gamma > 0$ and $q > 2$, then for any Lipschitz continuous function F on $L^2(\Lambda)$ we have*

$$\sup_{x \in L^2(\Lambda)} \big|P_t F(x) - \mu(F)\big| \leqslant C\,\mathrm{Lip}(F) t^{-\frac{1}{q-2}}, \quad t > 0, \tag{1.6}$$

where C is a constant.

Remark 1.1. (1) If $2 \geqslant p > \max\{1, \frac{2d}{d+2}\}$, then the embedding $H_0^{1,p}(\Lambda) \subseteq L^2(\Lambda)$ is compact according to the Rellich–Kondrachov theorem.

(2) If $B = 0$ and Dirac measure at 0 is the unique invariant measure of $\{P_t\}$, then by taking $F(x) = \|x\|_2$ in (1.6) we get the following algebraically decay estimate

$$\sup_{x \in L^2(\Lambda)} \big\|u_t(x)\big\|_2 \leqslant C t^{-\frac{1}{q-2}}, \quad t > 0,$$

where $u_t(x)$ is the solution of deterministic equation

$$\frac{\partial u}{\partial t} = \mathbf{div}\big(|\nabla u|^{p-2}\nabla u\big) - \gamma |u|^{q-2}u, \quad u_0 = x.$$

In order to use the coupling argument and Girsanov transformation, we need to assume that B is non-degenerate; that is, $Bx = 0$ implies $x = 0$. Then we can define the following intrinsic metric for $x \in H_0^{1,p}(\Lambda)$,

$$\|x\|_B := \begin{cases} \|y\|_2, & \text{if } y \in L^2(\Lambda),\ By = x, \\ \infty, & \text{otherwise.} \end{cases}$$

Theorem 1.2. *Suppose there exist constant $\sigma \geqslant \frac{4}{p}$ and $\xi > 0$ such that*

$$\|x\|_{1,p}^2 \cdot \|x\|_2^{\sigma-2} \geqslant \xi \|x\|_B^\sigma, \quad \forall x \in H_0^{1,p}(\Lambda), \tag{1.7}$$

then for any $t > 0$, P_t is strong Feller and for any positive $F \in \mathfrak{M}_b(L^2(\Lambda))$, $\alpha > 1$ and $x, y \in L^2(\Lambda)$ we have

$$\begin{aligned}(P_t F)^\alpha(y) \leqslant P_t F^\alpha(x) \exp\bigg[&\frac{\alpha-1}{4}\bigg(1 + 2te^{-(2b+1)t} + \|x\|_2^2 + \|y\|_2^2 + \frac{(\sigma+2)^2}{\sigma^2 t}\|x-y\|_2^2\bigg) \\ &+ \bigg(\frac{\sigma+2}{\sigma}\bigg)^{\sigma+1} \frac{[\alpha(\alpha+1)]^{\sigma/2} e^{(2b+1)(\sigma-2)t}}{4(p-1)\xi(\alpha-1)^{\sigma-1}t^\sigma}\|x-y\|_2^\sigma\bigg].\end{aligned} \tag{1.8}$$

Remark 1.2. (1) Note that if $\gamma = 0$ in (1.2), the Harnack inequality (1.8) still holds in the theorem above. In [19] the Harnack inequality has been obtained for the stochastic fast-diffusion equation, but we can not prove some strong contractivity for the transition semigroup because only linear perturbation considered in the drift. However, we can establish the ultraboundedness and compactness of the transition semigroup here if we have some high order absorption term ($\gamma > 0$) in the drift (see Theorem 1.3). This strong absorption term also plays important role in the convergence of the transition semigroup to its equilibrium (see Theorem 1.1).

(2) The estimate in right hand side of (1.8) comes from our coupling argument, which looks quite different with the known Gaussian type estimate in finite-dimensional case (cf. [36]). However, we know that the Gaussian type estimate in Harnack inequality is equivalent to some underlying curvature lower bound condition according to [38]. Hence it seems also reasonable to have this type of estimate (1.8) in the present case, which describes some worse long time behavior of the semigroup. We also refer to some estimate of similar form obtained in [4] for diffusion semigroup on manifolds with curvature unbounded below.

Theorem 1.3. *Assume* (1.7) *holds for* $\sigma \geqslant \frac{4}{p}$ *and* $\xi > 0$.

(i) $\{P_t\}$ *is* (*topologically*) *irreducible*; *i.e.* $P_t 1_U(\cdot) > 0$ *on* $L^2(\Lambda)$ *for every nonempty open set* U *and* $t > 0$. *And* $\{P_t\}$ *has a unique invariant measure* μ *with full support on* $L^2(\Lambda)$. *Moreover, for any probability measure* ν *on* $L^2(\Lambda)$ *we have*

$$\lim_{t\to\infty} \left\| P_t^* \nu - \mu \right\|_{\mathrm{var}} = 0,$$

where $\|\cdot\|_{\mathrm{var}}$ *is the variation norm on bounded* (*signed*) *Borel measures and* P_t^* *is the adjoint operator of* P_t.

(ii) *If* $p = 2$, *then we have* $\mu(e^{\varepsilon_0 \|\cdot\|_2^2}) < \infty$ *for some* $\varepsilon_0 > 0$. *Moreover,* P_t *is hyperbounded* (*i.e.* $\|P_t\|_{L^2(\mu)\to L^4(\mu)} < \infty$) *and compact for some* $t > 0$.

(iii) *If* $\gamma > 0$ *and* $q > \sigma$, *then* P_t *is ultrabounded* (*i.e.* $\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} < \infty$) *and compact on* $L^2(\mu)$ *for any* $t > 0$.

Remark 1.3. (1) Comparing with (ii) in Theorem 1.1, $\gamma = 0$ is allowed in (i) here. The uniqueness of invariant measures follows from the classical Doob theorem in this case.

(2) If $p = 2$, then (1.2) is the stochastic reaction–diffusion equation and the hyperbounded property of the transition semigroup also proved in [39, Theorem 1.2(3)]. However, if $\gamma > 0$ and $q > 2$ in (1.2), Theorem 1.3(iii) implies that the transition semigroups are ultrabounded and compact, which much stronger than the hyperbounded property.

This paper is organized as follows. The main theorems will be proved in the next section. In the last section, we present some explicit sufficient conditions such that (1.7) holds. Some concrete examples are also given for the applications of the main results.

2. Proofs of the theorems

2.1. Proof of Theorem 1.1

(i) The existence of invariant measure can be proved by the standard Krylov–Bogoliubov argument. Let

$$\mu_n := \frac{1}{n} \int_0^n \delta_0 P_t \,\mathrm{d}t, \quad n \geqslant 1,$$

where δ_0 is the Dirac measure at 0. Recall $X_t(y)$ is the solution of (1.2) with starting point y, then we have

$$\left\| X_t(x) - X_t(y) \right\|_2^2 \leqslant \|x - y\|_2^2, \quad \forall x, y \in L^2(\Lambda).$$

This implies that P_t is a Feller semigroup. Therefore, for the existence of the invariant measure, it is enough to verify the tightness of $\{\mu_n\colon n \geqslant 1\}$.

Since the embedding $H_0^{1,p}(\Lambda) \subseteq L^2(\Lambda)$ is compact, then $\|\cdot\|_{1,p}$ is a compact function on $L^2(\Lambda)$; i.e. for any constant K,

$$\left\{ x \in L^2(\Lambda)\colon \|x\|_{1,p} \leqslant K \right\}$$

is relatively compact in $L^2(\Lambda)$.

By the Itô formula (1.4) there exists a constant C such that

$$\mu_n\left(\|\cdot\|_{1,p}^p\right) = \frac{1}{n} \int_0^n \mathbb{E} \left\| X_t(0) \right\|_{1,p}^p \mathrm{d}t \leqslant C, \quad n \geqslant 1.$$

Hence $\{\mu_n\}$ is tight.

Combining with the Feller property, we know that the weak limit of a convergent subsequence of μ_n provides an invariant measure of P_t.

(ii) If $\gamma > 0$, then there exist positive constants c and C such that

$$\left\| X_t(x) - X_t(y) \right\|_2^2 \leqslant \|x - y\|_2^2 - c\gamma \int_0^t \left\| X_s(x) - X_s(y) \right\|_q^q \mathrm{d}s \leqslant \|x - y\|_2^2 - C \int_0^t \left\| X_s(x) - X_s(y) \right\|_2^q \mathrm{d}s.$$

Hence we have

$$\lim_{t\to\infty} \left\| X_t(x) - X_t(y) \right\|_2 = 0, \quad \forall x, y \in L^2(\Lambda).$$

This implies the uniqueness of invariant measures, which denoted by μ.

Now we need to prove the concentration property of invariant measure. (1.4) implies that there exists a constant C such that

$$\mu_n\big(\|\cdot\|_2^2\big)=\frac{1}{n}\int_0^n \mathbb{E}\big\|X_t(0)\big\|_2^2\,\mathrm{d}t\leqslant C,\quad n\geqslant 1.$$

Hence $\mu(\|\cdot\|_2^2)<\infty$, since μ is the weak limit of a subsequence of μ_n.

By (1.4) there also exists a constant C such that

$$\mathbb{E}\int_0^1 \big\|X_t(x)\big\|_{1,p}^p\,\mathrm{d}t\leqslant C\big(1+\|x\|_2^2\big),\quad \forall x\in L^2(\Lambda).$$

Therefore

$$\mu\big(\|\cdot\|_{1,p}^p\big)=\int\mu(\mathrm{d}x)\int_0^1\mathbb{E}\big(\big\|X_t(x)\big\|_{1,p}^p\big)\,\mathrm{d}t\leqslant C+C\int\|x\|_2^2\mu(\mathrm{d}x)<\infty.$$

If $\gamma>0$ and ε_0 is small enough, then by Itô's formula

$$\begin{aligned}
e^{\varepsilon_0\|X_t\|_2^q}&\leqslant e^{\varepsilon_0\|x\|_2^q}+\int_0^t\big(c-2\gamma\|X_s\|_q^q+qb\varepsilon_0\|X_s\|_2^q\big)\frac{q\varepsilon_0}{2}\|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\,\mathrm{d}s+q\varepsilon_0\int_0^t\|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\langle X_s,B\,\mathrm{d}W_s\rangle\\
&\leqslant e^{\varepsilon_0\|x\|_2^q}+\int_0^t\big(c-\gamma\|X_s\|_q^q\big)\frac{q\varepsilon_0}{2}\|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\,\mathrm{d}s+q\varepsilon_0\int_0^t\|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\langle X_s,B\,\mathrm{d}W_s\rangle\\
&\leqslant e^{\varepsilon_0\|x\|_2^q}+\int_0^t\big(c_1-c_2e^{\varepsilon_0\|X_s\|_2^q}\big)\,\mathrm{d}s+q\varepsilon_0\int_0^t\|X_s\|_2^{q-2}e^{\varepsilon_0\|X_s\|_2^q}\langle X_s,B\,\mathrm{d}W_s\rangle
\end{aligned}\tag{2.1}$$

for some positive constants c, c_1 and c_2. Therefore

$$\mu_n\big(e^{\varepsilon_0\|\cdot\|_2^q}\big)=\frac{1}{n}\int_0^n\mathbb{E}e^{\varepsilon_0\|X_t(0)\|_2^q}\,\mathrm{d}t\leqslant\frac{1}{c_2n}+\frac{c_1}{c_2}.$$

Hence we have $\mu(e^{\varepsilon_0\|\cdot\|_2^q})<\infty$ for some $\varepsilon_0>0$.

(iii) If $\gamma>0$ and $q=2$, then we have

$$\big\|X_t(x)-X_t(y)\big\|_2^2\leqslant\big\|X_s(x)-X_s(y)\big\|_2^2-2\gamma\int_s^t\big\|X_u(x)-X_u(y)\big\|_2^2\,\mathrm{d}u,\quad t\geqslant s\geqslant 0.$$

It implies

$$\big\|X_t(x)-X_t(y)\big\|_H^2\leqslant\|x-y\|_2^2e^{-2\gamma t},\quad\forall x,y\in L^2(\Lambda).$$

Hence, for any bounded Lipschitz function F on $L^2(\Lambda)$, we have

$$\big|P_tF(x)-\mu(F)\big|\leqslant\int_{L^2(\Lambda)}\mathbb{E}\big|F\big(X_t(x)\big)-F\big(X_t(y)\big)\big|\mu(\mathrm{d}y)\leqslant\mathrm{Lip}(F)e^{-\gamma t}\int\|x-y\|_2\mu(\mathrm{d}y).$$

Since $\mu(\|\cdot\|_2^2)<\infty$, there exists a constant $C>0$ such that

$$\big|P_tF(x)-\mu(F)\big|\leqslant\mathrm{Lip}(F)e^{-\gamma t}\big(\|x\|_2+C\big),\quad\forall x\in L^2(\Lambda).$$

(iv) Since $\gamma>0$ and $q>2$, there exists a constant c such that

$$\big\|X_t(x)-X_t(y)\big\|_2^2\leqslant\big\|X_s(x)-X_s(y)\big\|_2^2-c\int_s^t\big\|X_u(x)-X_u(y)\big\|_2^q\,\mathrm{d}u,\quad t\geqslant s\geqslant 0.$$

For $\varepsilon > 0$ we define

$$h_t = \left\{(\varepsilon + \|x-y\|_2)^{2-q} + \frac{(q-2)c_0}{2}t\right\}^{-\frac{2}{q-2}}.$$

It is easy to show that h_t solves the equation

$$h_t' = -ch_t^{\frac{q}{2}}, \qquad h_s = \left(\left\|X_s(x) - X_s(y)\right\|_2 + \varepsilon\right)^2. \tag{2.2}$$

Then by a standard comparison argument we have

$$\left\|X_t(x) - X_t(y)\right\|_2^2 \leqslant h_t \leqslant Ct^{-\frac{2}{q-2}}, \tag{2.3}$$

where $C > 0$ is a constant. In fact, we define

$$\varphi_t := h_t - \left\|X_t(x) - X_t(y)\right\|_2^2, \qquad \tau := \inf\{t \geqslant 0\colon \varphi_t < 0\}.$$

If $\tau < +\infty$, then we know $\varphi_\tau \leqslant 0$ by the continuity.

By the mean-value theorem there exists a constant $C > 0$,

$$\varphi_t \geqslant \varphi_0 - c\int_0^t \left(h_s^{\frac{q}{2}} - \left\|X_s(x) - X_s(y)\right\|_2^q\right) \mathrm{d}s \geqslant \varepsilon^2 - C\int_0^t \varphi_s \,\mathrm{d}s, \quad 0 \leqslant t \leqslant \tau.$$

Hence by the Gronwall lemma we have

$$\varphi_\tau \geqslant \varepsilon^2 e^{-C\tau} > 0,$$

which is contradict to $\varphi_\tau \leqslant 0$. Therefore, (2.3) holds.

Therefore, for any bounded Lipschitz function F on $L^2(\Lambda)$, we have for any $x \in L^2(\Lambda)$,

$$\left|P_tF(x) - \mu(F)\right| \leqslant \int_{L^2(\Lambda)} \mathbb{E}\left|F\left(X_t(x) - F\left(X_t(y)\right)\right)\right|\mu(\mathrm{d}y) \leqslant C\,\mathrm{Lip}(F)t^{-\frac{1}{q-2}}.$$

Hence (1.6) holds. □

2.2. Proof of Theorem 1.2

The main techniques is the coupling argument and Girsanov transformation. It seems that [21] is the first paper using a coupling method to prove the uniqueness of invariant measures and the mixing property for a stochastic partial differential equation. Recently, the coupling argument have been used to prove the ergodicity and exponential convergence to equilibrium for the Navier–Stokes equation driven by very degenerate noises [16,17,23,24]. It has also been employed in [13] for stochastic reaction–diffusion equations, in [7] for stochastic Burgers equations and in [25] for stochastic Ginzburg–Landau equations. But here the coupling is constructed in a quite different way.

In order to make the proof easier to follow, we first explain the main idea and steps of the proof. For any two fixed points $x, y \in L^2(\Lambda)$ and fixed time T, let $X_t(x)$ and $X_t(y)$ denote the solution of (1.2) with starting points x, y respectively. In order to force the two marginal processes to meet before the given time T, we need to add an appropriate drift to one marginal process (e.g. $X_t(y)$). We may denote the new modified process by $Y_t(y)$, which should satisfy the following conditions:

(i) $X_T(x) = Y_T(y)$, a.s.
(ii) $Y_t(y)$ solves the equation

$$\mathrm{d}Y_t = \left\{\mathbf{div}\left(|\nabla Y_t|^{p-2}\nabla Y_t\right) - \gamma|Y_t|^{q-2}Y_t\right\}\mathrm{d}t + B\,\mathrm{d}\tilde{W}_t, \quad Y_0 = y$$

for another cylindrical Wiener process $\tilde{W}_t$ on $L^2(\Lambda)$ under a weighted probability measure $R\mathbb{P}$, where the density R will be induced by some Girsanov transformation later.

Since $Y_t(y)$ (under the weighted probability measure) has the same distribution with $X_t(y)$, then for any bounded positive measurable function on $L^2(\Lambda)$ we have

$$\begin{aligned}(P_TF)^\alpha(y) &= \left(\mathbb{E}F\left(X_T(y)\right)\right)^\alpha = \left(\mathbb{E}RF\left(Y_T(y)\right)\right)^\alpha = \left(\mathbb{E}RF\left(X_T(x)\right)\right)^\alpha \leqslant \mathbb{E}F\left(X_T(x)\right)^\alpha\left(\mathbb{E}R^{\alpha/(\alpha-1)}\right)^{\alpha-1}\\ &= P_TF^\alpha(x)\left(\mathbb{E}R^{\alpha/(\alpha-1)}\right)^{\alpha-1}, \quad \alpha > 1,\end{aligned} \tag{2.4}$$

which implies the desired Harnack inequality provided $\mathbb{E}R^{\alpha/(\alpha-1)} < \infty$.

Now we can construct the coupling process $Y_t(y)$. For $\varepsilon \in (0,1)$ and $\beta \in \mathbf{C}([0,\infty);\mathbb{R}^+)$, consider the following equation of Y_t,

$$dY_t = \left\{ \mathbf{div}\big(|\nabla Y_t|^{p-2}\nabla Y_t\big) - \gamma |Y_t|^{q-2}Y_t + \frac{\beta_t(X_t - Y_t)}{\|X_t - Y_t\|_2^{\varepsilon}} \mathbf{1}_{\{t<\tau\}} \right\} dt + B\, dW_t, \quad Y_0 = y, \tag{2.5}$$

where $X_t := X_t(x)$ and $\tau := \inf\{t \geqslant 0\colon\ X_t = Y_t\}$ is the coupling time.

Lemma 2.1. *If $\varepsilon \in [2-p, 1)$, then there exists a unique strong solution Y_t of (2.5). Moreover, we have $X_t = Y_t$ for all $t \geqslant \tau$.*

Proof. According to [15,40], We only have to verify that the coefficients of (2.5) satisfy the monotone and coercive conditions $(A1)$–$(A4)$. Let

$$A(t,x) := \frac{X_t - x}{\|X_t - x\|_2^{\varepsilon}} \mathbf{1}_{\{t<\tau\}}.$$

Since $\sup_{t\in[0,T]} \mathbb{E}\|X_t\|_2^2 < \infty$, Thus, $A(t,x) \in L^2(\Lambda)$ with

$$\big\|A(t,x)\big\|_2 = \|X_t - x\|_2^{1-\varepsilon}, \quad x \in L^2(\Lambda).$$

Therefore, $(A1)$, $(A3)$ hold, and $(A4)$ also holds since $\varepsilon \geqslant 2-p$.

To verify $(A2)$, it is enough to prove the following monotonicity

$$\big\langle A(t,x) - A(t,y), x-y \big\rangle \leqslant 0 \quad \text{on } \Omega,\ x, y \in L^2(\Lambda). \tag{2.6}$$

By the symmetry, for a fixed $\omega \in \Omega$ it is sufficient to verify (2.6) for $x, y \in L^2(\Lambda)$ with

$$\|X_t - x\|_2 \leqslant \|X_t - y\|_2. \tag{2.7}$$

(i) If $\|X_t - x\|_2 \geqslant \|x - y\|_2$, then by (2.7) and the mean valued theorem we have

$$\begin{aligned}
\big\langle A(t,x) - A(t,y), x-y \big\rangle &= -\frac{\|x-y\|_2^2}{\|X_t - x\|_2^{\varepsilon}} + \frac{\|X_t - y\|_2^{\varepsilon} - \|X_t - x\|_2^{\varepsilon}}{\|X_t - y\|_2^{\varepsilon}\|X_t - x\|_2^{\varepsilon}} \langle X_t - y, x-y\rangle \\
&\leqslant -\frac{\|x-y\|_2^2}{\|X_t - x\|_2^{\varepsilon}} + \frac{\varepsilon\|X_t - y\|_2^{1-\varepsilon}\|x-y\|_2^2}{\|X_t - x\|_2} \\
&\leqslant -\frac{\|x-y\|_2^2}{\|X_t - x\|_2^{\varepsilon}} + \frac{\varepsilon 2^{-\varepsilon}(\|X_t - x\|_2^{1-\varepsilon} + \|x-y\|_2^{1-\varepsilon})\|x-y\|_2^2}{\|X_t - x\|_2} \\
&\leqslant -\frac{(1-\varepsilon 2^{1-\varepsilon})\|x-y\|_2^2}{\|X_t - x\|_2^{\varepsilon}} \leqslant 0,
\end{aligned}$$

where in the third step we use the inequality for $r > 0$,

$$(a+b)^r \leqslant 2^{r-1}\big(a^r + b^r\big), \quad a, b \geqslant 0.$$

(ii) If $\|X_t - x\|_2 \leqslant \|x - y\|_2$, then (2.6) can be proved by the similar argument.

Therefore, according to [40, Theorem 3.6], (2.5) also has a unique strong solution $Y_t(y)$. Moreover, we have

$$\|X_t - Y_t\|_2 \leqslant \|X_s - Y_s\|_2, \quad t \geqslant s \geqslant 0.$$

By the definition of τ, we have $X_t = Y_t$ for $t \geqslant \tau$. $\square$

Now take

$$\zeta_t := \frac{\beta_t B^{-1}(X_t - Y_t)}{\|X_t - Y_t\|_2^{\varepsilon}} \mathbf{1}_{\{t<\tau\}},$$

we can rewrite Eq. (2.5) into

$$dY_t = \big(\mathbf{div}\big(|\nabla Y_t|^{p-2}\nabla Y_t\big) - \gamma|Y_t|^{q-2}Y_t\big)\, dt + B(dW_t + \zeta_t\, dt), \quad Y_0 = y.$$

Therefore, to verify (i) and (ii), we only need to choose $\varepsilon \in [2-p, 1)$ and β such that

(a) $\tau \leqslant T$, a.s.;
(b) $\mathbb{E}\exp[\frac{1}{2}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t] < \infty$.

Note (b) implies that $\tilde{W}_t := W_t + \int_0^t \zeta_s\,\mathrm{d}s$ is a cylindrical Wiener process under $R\mathbb{P}$, where

$$R = \exp\left[-\int_0^T \langle \zeta_t, \mathrm{d}W_t\rangle - \frac{1}{2}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t\right]. \tag{2.8}$$

Lemma 2.2. *If β satisfies $\int_0^T \beta_t\,\mathrm{d}t \geqslant \frac{1}{\varepsilon}\|x-y\|_2^{\varepsilon}$, then $\tau \leqslant T$, a.s.*

Proof. We can show that for any u, v in $H_0^{1,p}(\Lambda)$ (see Lemma 3.1),

$$\left\langle \mathbf{div}\left(|\nabla u|^{p-2}\nabla u\right) - \mathbf{div}\left(|\nabla v|^{p-2}\nabla v\right), u-v\right\rangle_0 \leqslant -(p-1)\mathbf{m}\left(|\nabla u - \nabla v|^2\left(|\nabla u| \vee |\nabla v|\right)^{p-2}\right),$$

where $\mathbf{m}$ is the Lebesgue measure on $\mathbb{R}^d$ and $\langle\cdot,\cdot\rangle_0$ denote the dualization between $H_0^{1,p}(\Lambda)$ and its dual space.

By the Itô formula there exists a constant $c > 0$ such that

$$\begin{aligned}\|X_t - Y_t\|_2^2 \leqslant \|X_s - Y_s\|_2^2 - 2(p-1)\int_s^t \mathbf{m}\left(|\nabla X_u - \nabla Y_u|^2\left(|\nabla X_u| \vee |\nabla Y_u|\right)^{p-2}\right)\mathrm{d}u \\ - 2\int_s^t \beta_t\|X_u - Y_u\|_2^{2-\varepsilon}\mathbf{1}_{\{u<\tau\}}\,\mathrm{d}u - c\gamma\int_s^t \|X_u - Y_u\|_q^q\,\mathrm{d}u.\end{aligned} \tag{2.9}$$

This implies

$$\|X_t - Y_t\|_2^{\varepsilon} \leqslant \|x-y\|_2^{\varepsilon} - \varepsilon\int_0^t \beta_u\,\mathrm{d}u, \quad t \leqslant \tau \wedge T.$$

If $T < \tau(\omega)$ for some ω, then by the assumption we have

$$\left\|X_T(\omega) - Y_T(\omega)\right\|_2^{\varepsilon} \leqslant \|x-y\|_2^{\varepsilon} - \varepsilon\int_0^T \beta_t\,\mathrm{d}t \leqslant 0.$$

Hence $X_T(\omega) = Y_T(\omega)$, which is contradictory to $T < \tau(\omega)$.

Therefore, we have $\tau \leqslant T$, a.s. $\square$

In order to verify (b), first we need to have the following a priori estimate:

Lemma 2.3. *We have*

$$\mathbb{E}\exp\left[\lambda_T\int_0^T \|X_t\|_{1,p}^p\,\mathrm{d}t\right] \leqslant \exp\left[\|x\|_2^2 + \int_0^T be^{-2bt}\,\mathrm{d}t\right], \tag{2.10}$$

$$\mathbb{E}\exp\left[\lambda_T\int_0^T \|Y_t\|_{1,p}^p\,\mathrm{d}t\right] \leqslant \exp\left[\|y\|_2^2 + \int_0^T be^{-(2b+1)t}\,\mathrm{d}t + \|x-y\|_H^{2(1-\varepsilon)}\int_0^T \beta_t^2 e^{-(2b+1)t}\,\mathrm{d}t\right], \tag{2.11}$$

where $\lambda_T = 2e^{-(2b+1)T}$ and $b = \|B\|_{\mathrm{HS}}^2$.

Proof. By the Itô formula (1.4) we have

$$\mathrm{e}^{-2bT}\|X_T\|_2^2 \leqslant \|x\|_2^2 + \int_0^T \mathrm{e}^{-2bt}\left(b - 2\|X_t\|_{1,p}^p - 2b\|X_t\|_2^2\right)\mathrm{d}t + 2\int_0^T \mathrm{e}^{-2bt}\langle X_t, B\,\mathrm{d}W_t\rangle.$$

This implies that

$$2e^{-2bT}\int_0^T \|X_t\|_{1,p}^p\,dt \leqslant \|x\|_2^2+\int_0^T be^{-2bt}\,dt+M_T-\int_0^T 2be^{-2bt}\|X_t\|_2^2\,dt$$

where $M_T=2\int_0^T e^{-2bt}\langle X_t, B\,dW_t\rangle$.

It is easy to check from (1.4) and (1.3) that $\{M_t\}$ is a martingale. By taking $\lambda_T=2e^{-(2b+1)T}$ we obtain

$$\mathbb{E}\exp\left[\lambda_T\int_0^T \|X_t\|_{1,p}^p\,dt\right]\leqslant \exp\left[\int_0^T be^{-2bt}\,dt+\|x\|_H^2\right]\mathbb{E}\exp\left[M_T-\int_0^T 2be^{-2bt}\|X_t\|_2^2\,dt\right].$$

Since $\langle M\rangle_t\leqslant\int_0^T 4be^{-4bt}\|X_t\|_2^2\,dt$ and $\mathbb{E}\exp[M_t-\frac{1}{2}\langle M\rangle_t]=1$, then

$$\mathbb{E}\exp\left[M_T-\int_0^T 2be^{-2bt}\|X_t\|_2^2\,dt\right]\leqslant 1.$$

Hence (2.10) holds.

Since (2.9) implies that

$$\|X_t-Y_t\|_2^2\leqslant\|x-y\|_2^2,\quad t\geqslant 0,$$

then by the Itô formula we have

$$\begin{aligned}
e^{-(2b+1)T}\|Y_T\|_2^2 &\leqslant \|y\|_2^2+\int_0^T e^{-(2b+1)t}\big[b-2\|Y_t\|_{1,p}^p-(2b+1)\|Y_t\|_2^2+2\|Y_t\|_2\beta_t\|X_t-Y_t\|_2^{1-\varepsilon}\big]\,dt+M_T'\\
&\leqslant \|y\|_2^2+\int_0^T e^{-(2b+1)t}\big[b-2\|Y_t\|_{1,p}^p-2b\|Y_t\|_2^2+\beta_t^2\|x-y\|_2^{2(1-\varepsilon)}\big]\,dt+M_T',
\end{aligned}$$

where $M_t':=\int_0^t 2e^{-(2b+1)s}\langle Y_s, B\,dW_s\rangle$ is a martingale. This implies

$$\begin{aligned}
2e^{-(2b+1)T}\int_0^T \|Y_t\|_{1,p}^p\,dt &\leqslant \|y\|_2^2+\int_0^T be^{-(2b+1)t}\,dt+\|x-y\|_2^{2(1-\varepsilon)}\int_0^T \beta_t^2e^{-(2b+1)t}\,dt\\
&\quad+M_T'-\int_0^T 2be^{-(2b+1)t}\|Y_t\|_2^2\,dt.
\end{aligned}$$

Therefore, by the similar argument and noting that

$$\langle M'\rangle_T\leqslant\int_0^T 4be^{-2(2b+1)t}\|Y_t\|_2^2\,dt,$$

we obtain (2.11). □

Proof of the Theorem 1.2. From now on, we take $\varepsilon=\frac{\sigma}{\sigma+2}$ and

$$\beta_t=c\big(2(p-1)\varepsilon\xi\big)^{1/\sigma},\quad c=\frac{\|x-y\|_2^{\varepsilon}}{\varepsilon(2(p-1)\varepsilon\xi)^{\frac{1}{\sigma}}T}.$$

It's easy to see that $\varepsilon\in[2-p,1)$. Therefore, according to Lemma 2.1 and Lemma 2.2, there exists a unique solution Y_t of (2.5) such that the coupling time $\tau\leqslant T$, a.s.

By the Hölder inequality we have

$$\|X_t-Y_t\|_{1,p}^2\leqslant\mathbf{m}\big(|\nabla X_t-\nabla Y_t|^2\big(|\nabla X_t|\vee|\nabla Y_t|\big)^{p-2}\big)\cdot\big(\mathbf{m}\big[\big(|\nabla X_t|\vee|\nabla Y_t|\big)^p\big]\big)^{\frac{2-p}{p}}.$$

Let $f_t := (\mathbf{m}[(|\nabla X_t| \vee |\nabla Y_t|)^p])^{\frac{2-p}{p}}$, then by the Itô formula, (2.9) and (1.7),

$$\begin{aligned}
\mathrm{d}\big(\|X_t - Y_t\|_2^2\big)^{\varepsilon} &\leqslant -2(p-1)\varepsilon \|X_t - Y_t\|_2^{2(\varepsilon-1)} \mathbf{m}\big(|\nabla X_t - \nabla Y_t|^2 (|\nabla X_t| \vee |\nabla Y_t|)^{p-2}\big)\,\mathrm{d}t \\
&\leqslant -2(p-1)\varepsilon \|X_t - Y_t\|_2^{2(\varepsilon-1)} \frac{\|X_t - Y_t\|_{1,p}^2}{(\mathbf{m}[(|\nabla X_t| \vee |\nabla Y_t|)^p])^{\frac{2-p}{p}}}\,\mathrm{d}t \\
&\leqslant -2(p-1)\varepsilon\xi \frac{\|X_t - Y_t\|_B^{\sigma}}{\|X_t - Y_t\|_2^{\sigma-2\varepsilon} f_t}\,\mathrm{d}t \\
&= -2(p-1)\varepsilon\xi \frac{\|X_t - Y_t\|_B^{\sigma}}{\|X_t - Y_t\|_2^{\sigma\varepsilon} f_t}\,\mathrm{d}t \\
&= -\frac{\beta_t^{\sigma} \|X_t - Y_t\|_B^{\sigma}}{c^{\sigma} \|X_t - Y_t\|_2^{\sigma\varepsilon} f_t}\,\mathrm{d}t.
\end{aligned}$$

Combining this with the Young inequality we obtain

$$\begin{aligned}
\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t &= \int_0^T \frac{\beta_t^2 \|X_t - Y_t\|_B^2}{\|X_t - Y_t\|_2^{2\varepsilon}}\,\mathrm{d}t \\
&\leqslant \left(\int_0^T f_t^{\frac{2}{\sigma-2}}\,\mathrm{d}t\right)^{\frac{\sigma-2}{\sigma}} \left(\int_0^T \frac{\beta_t^{\sigma} \|X_t - Y_t\|_B^{\sigma}}{\|X_t - Y_t\|_2^{\sigma\varepsilon} f_t}\,\mathrm{d}t\right)^{\frac{2}{\sigma}} \\
&\leqslant \left(\int_0^T f_t^{\frac{2}{\sigma-2}}\,\mathrm{d}t\right)^{\frac{\sigma-2}{\sigma}} \big(c^{\sigma} \|x - y\|_2^{2\varepsilon}\big)^{\frac{2}{\sigma}} \\
&\leqslant \lambda \int_0^T f_t^{\frac{2}{\sigma-2}}\,\mathrm{d}t + \lambda^{(2-\sigma)/2} c^{\sigma} \|x - y\|_2^{2\varepsilon}, \quad \lambda > 0. \qquad (2.12)
\end{aligned}$$

Since $\sigma \geqslant \frac{4}{p}$ implies $\frac{2}{\sigma-2} \leqslant \frac{p}{2-p}$, then we have

$$f_t^{\frac{2}{\sigma-2}} \leqslant \mathbf{m}\big(|\nabla X_t|^p \vee |\nabla Y_t|^p\big)^{\frac{2(2-p)}{(\sigma-2)p}} \leqslant 1 + \|X_t\|_{1,p}^p + \|Y_t\|_{1,p}^p.$$

Let $\lambda = \lambda_T$ in (2.12), then by Lemma 2.3 it's easy to show (b) holds; i.e.

$$\mathbb{E}\exp\left[\frac{1}{2}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t\right] < \infty.$$

Now combining (2.4), (2.8) and Hölder's inequality we have

$$\begin{aligned}
\big(P_T F(y)\big)^{\alpha} &\leqslant P_T F^{\alpha}(x) \big(\mathbb{E}R^{\alpha/(\alpha-1)}\big)^{\alpha-1} \\
&= P_T F^{\alpha}(x) \left\{\mathbb{E}\exp\left[\frac{\alpha}{\alpha-1}\int_0^T \langle \zeta_t, \mathrm{d}W_t\rangle - \frac{\alpha}{2(\alpha-1)}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t\right]\right\}^{\alpha-1} \\
&\leqslant P_T F^{\alpha}(x) \left\{\mathbb{E}\exp\left[\frac{2\alpha}{\alpha-1}\int_0^T \langle \zeta_t, \mathrm{d}W_t\rangle - \frac{2\alpha^2}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t\right]\right\}^{\frac{\alpha-1}{2}} \\
&\quad \cdot \left\{\mathbb{E}\exp\left[\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t\right]\right\}^{\frac{\alpha-1}{2}} \\
&\leqslant P_T F^{\alpha}(x) \left\{\mathbb{E}\exp\left[\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t\right]\right\}^{\frac{\alpha-1}{2}}. \qquad (2.13)
\end{aligned}$$

Taking $\lambda = \frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}$ in (2.12), by Lemma 2.3 we obtain that

$$\begin{aligned}
&\mathbb{E}\exp\left[\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t\right] \\
&\leqslant \mathbb{E}\exp\left[\frac{\lambda_T}{2}\int_0^T \left(1+\|X_t\|_{1,p}^p+\|Y_t\|_{1,p}^p\right)\mathrm{d}t + \frac{\alpha(\alpha+1)}{(\alpha-1)^2}\left(\frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}\right)^{\frac{2-\sigma}{2}} c^\sigma\|x-y\|_2^{2\varepsilon}\right] \\
&\leqslant \exp\left[\frac{1}{2}\left(\lambda_T T+\|x\|_2^2+\|y\|_2^2+2\int_0^T be^{-2bt}\,\mathrm{d}t+\|x-y\|_2^{2(1-\varepsilon)}\int_0^T \beta_t^2 e^{-(2b+1)t}\,\mathrm{d}t\right)\right. \\
&\qquad \left.+\frac{\alpha(\alpha+1)}{(\alpha-1)^2}\left(\frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}\right)^{\frac{2-\sigma}{2}} c^\sigma\|x-y\|_2^{2\varepsilon}\right].
\end{aligned} \tag{2.14}$$

Combing this with (2.13) we have

$$\begin{aligned}
\left(P_TF(y)\right)^\alpha &\leqslant P_TF^\alpha(x)\exp\left[\frac{\alpha-1}{4}\left(1+\lambda_T T+\|x\|_2^2+\|y\|_2^2++\|x-y\|_2^{2(1-\varepsilon)}\int_0^T \beta_t^2 e^{-(2b+1)t}\,\mathrm{d}t\right)\right. \\
&\qquad \left.+\frac{\alpha(\alpha+1)}{2(\alpha-1)}\left(\frac{\lambda_T(\alpha-1)^2}{2\alpha(\alpha+1)}\right)^{\frac{2-\sigma}{2}} c^\sigma\|x-y\|_2^{2\varepsilon}\right].
\end{aligned} \tag{2.15}$$

Then the desired result (1.8) follows by using the definition of β_t and c.

Finally, since

$$\left|P_TF(y)-P_TF(x)\right| = \left|\mathbb{E}(R-1)F(X_T)\right| \leqslant \|F\|_\infty \mathbb{E}|R-1|,$$

and since due to (2.13) R is uniformly integrable for fixed x and $\{y\colon \|x-y\|_2 \leqslant 1\}$, then by the dominated convergence theorem we have for any bounded measurable function F on $L^2(\Lambda)$,

$$\lim_{y\to x}\left|P_TF(y)-P_TF(x)\right| \leqslant \|F\|_\infty \lim_{y\to x}\mathbb{E}|R-1| = \|F\|_\infty \mathbb{E}\left(\lim_{y\to x}|R-1|\right) = 0$$

where the last equality follows from (2.12).

Therefore, P_T is strong Feller operator. Now the proof is complete. □

2.3. *Proof of Theorem 1.3*

(i) Since B is a Hilbert–Schmidt operator, we know that $\|\cdot\|_B$ is a compact function on $L^2(\Lambda)$; i.e. $\{x\in L^2(\Lambda)\colon \|x\|_B \leqslant N\}$ is relatively compact in $L^2(\Lambda)$ for any $N>0$.

Then (1.7) implies that for any $N>0$,

$$\left\{x\in L^2(\Lambda)\colon \|x\|_{1,p}\leqslant N\right\}$$

is also relatively compact in $L^2(\Lambda)$; i.e. the embedding $H_0^{1,p}(\Lambda)\subseteq L^2(\Lambda)$ is compact.

Therefore, the existence of invariant measure follows from Theorem 1.1(i). The full support of the invariant measure follows from the Harnack inequality (1.8) by repeating the proof of Theorem 1.2(1) in [39].

By (1.8) we have for any $x, x_0 \in L^2(\Lambda)$,

$$(P_t1_U)^p(x_0) \leqslant P_t1_U(x)\exp\left[C\left(1+\|x_0\|_2^2+\|x\|_2^2+\|x-x_0\|_2^2+\|x-x_0\|_2^\sigma\right)\right],$$

where $C>0$ is some constant.

Therefore, in order to prove the irreducibility, one only has to show for any given nonempty open set U and $t>0$, there exists $x_0\in H$ such that $P_t1_U(x_0)>0$.

Since μ has full support, then we have

$$\int_H P_t1_U(x)\mu(\mathrm{d}x) = \int_H 1_U(x)\mu(\mathrm{d}x) = \mu(U) > 0. \tag{2.16}$$

Hence $P_t1_U(\cdot)$ cannot be the zero function. Therefore, there exists at least one $x_0\in H$ such that $P_t1_U(x_0)>0$.

Since $\{P_t\}$ is also strong Feller, then the uniqueness of invariant measures follows from the classical Doob theorem [10].

Note that the solution has continuous paths on H, the other assertions follow from the general result in ergodic theory, we refer to [32, Theorem 2.2 and Proposition 2.5] or [22].

(ii) If $p=2$ and ε_0 is small enough, then by Itô's formula and Poincaré's inequality

$$\begin{aligned} e^{\varepsilon_0\|X_t\|_2^2} &\leqslant e^{\varepsilon_0\|x\|_2^2} + \int_0^t \big(c - 2\|X_s\|_{1,2}^2 + 2b\varepsilon_0\|X_s\|_2^2\big)\varepsilon_0 e^{\varepsilon_0\|X_s\|_2^2}\,\mathrm{d}s + 2\varepsilon_0\int_0^t e^{\varepsilon_0\|X_s\|_2^2}\langle X_s, B\,\mathrm{d}W_s\rangle \\ &\leqslant e^{\varepsilon_0\|x\|_2^2} + \int_0^t \big(c_1 - c_2 e^{\varepsilon_0\|X_s\|_2^2}\big)\,\mathrm{d}s + 2\varepsilon_0\int_0^t e^{\varepsilon_0\|X_s\|_2^2}\langle X_s, B\,\mathrm{d}W_s\rangle \end{aligned} \tag{2.17}$$

for some positive constants c, c_1 and c_2.

Hence by the same argument in Theorem 1.1(ii) we have $\mu(e^{\varepsilon_0\|\cdot\|_2^2}) < \infty$.

If $p=2$, one can just repeat the proof for (1.8) (Lemma 2.3 can be omitted) and (2.12) turns to be

$$\int_0^T \|\zeta_t\|_2^2\,\mathrm{d}t = \int_0^T \frac{\beta_t^2\|X_t - Y_t\|_B^2}{\|X_t - Y_t\|_2^{2\varepsilon}}\,\mathrm{d}t \leqslant T^{\frac{\sigma-2}{\sigma}}\big(c^\sigma\|x-y\|_2^{2\varepsilon}\big)^{\frac{2}{\sigma}}.$$

Hence we can get the following Harnack inequality

$$(P_tF)^\alpha(y) \leqslant P_tF^\alpha(x)\exp\left[\frac{C\alpha(\alpha+1)}{(\alpha-1)t^{(\sigma+2)/\sigma}}\|x-y\|_2^2\right]$$

where C is a constant depending on σ and ξ.

Since $\mu(e^{\varepsilon_0\|\cdot\|_2^2}) < \infty$, by the same argument as in [39, Theorem 1.3(2)] one can obtain the hyperbounded and compact property of P_t for some large $t>0$.

(iii) If $\gamma > 0$ and $q > \sigma$, then by Itô's formula and (2.1) we have for small $\varepsilon_0 > 0$,

$$e^{\varepsilon_0\|X_t\|_2^q} \leqslant e^{\varepsilon_0\|x\|_2^q} + \int_0^t \big(c_2 - c_1\|X_s\|_2^{2q-2}e^{\varepsilon_0\|X_s\|_2^q}\big)\,\mathrm{d}s + M_t' \tag{2.18}$$

where c_1 and c_2 are some positive constants and M' is a local martingale. By Jensen's inequality we have

$$\mathbb{E}e^{\varepsilon_0\|X_t\|_2^q} \leqslant e^{\varepsilon_0\|x\|_2^q} + c_2t - c_1\varepsilon_0^{-(2q-2)/q}\int_0^t \mathbb{E}e^{\varepsilon_0\|X_s\|_2^q}\big(\log\mathbb{E}e^{\varepsilon_0\|X_s\|_2^q}\big)^{\frac{2q-2}{q}}\,\mathrm{d}s.$$

Let $h(t)$ solve the equation

$$h'(t) = c_2 - c_1\varepsilon_0^{-(2q-2)/q}h(t)\big\{\log h(t)\big\}^{(2q-2)/q}, \qquad h(0) = \exp\left[\varepsilon_0\left(\|x\|_2^q + \left(\frac{4c_2}{c_1}\right)^{\frac{q}{2q-2}}\right)\right]. \tag{2.19}$$

Then by the standard comparison argument we know

$$\mathbb{E}e^{\varepsilon_0\|X_t(x)\|_2^q} \leqslant h(t). \tag{2.20}$$

Note that $\frac{2q-2}{q} > 1$. Combining (2.19) and (2.20), we get

$$\mathbb{E}e^{\varepsilon_0\|X_t(x)\|_2^q} \leqslant \exp\big[c_0\big(1 + t^{-q/(q-2)}\big)\big], \quad t>0,\ x\in L^2(\Lambda) \tag{2.21}$$

for some constant $c_0 > 0$.

Let $f \in L^2(\mu)$ with $\mu(f^2) = 1$, (1.8) implies that there exists $c_t > 0$ depending on t (which may change from line to line) such that

$$(P_tf)^2(x)\exp\big[c_t\big(1 + \|x\|_2^2 + \|y\|_2^2 + \|x-y\|_2^2 + \|x-y\|_2^\sigma\big)\big] \leqslant P_tf^2(y), \quad x,y\in L^2(\Lambda),\ t>0. \tag{2.22}$$

Integrating for both sides w.r.t. $\mu(\mathrm{d}y)$, we obtain

$$(P_tf)^2(x) \leqslant \frac{1}{\mu(B(0,1))}\exp\big[c_t\big(1 + \|x\|_2^2 + \|x\|_2^\sigma\big)\big], \quad x\in L^2(\Lambda),\ t>0, \tag{2.23}$$

where $B(0,1) := \{y\in L^2(\Lambda)\colon\ \|y\|_2 \leqslant 1\}$ has positive mass with respect to μ. Hence we have

$$\|P_t f\|_\infty = \|P_{t/2}P_{t/2}f\|_\infty \leqslant c \sup_{x\in L^2(\Lambda)} \mathbf{E}\exp\bigl[c_t\bigl(1+\|X_{t/2}(x)\|_2^2+\|X_{t/2}(x)\|_2^\sigma\bigr)\bigr], \quad t>0 \tag{2.24}$$

for some $c>0$. Since $q>\sigma$, by Young's inequality there exists $C_t>0$ such that

$$c_t\bigl(1+u^2+u^\sigma\bigr) \leqslant C_t+\varepsilon_0 u^q, \quad u>0.$$

Therefore, we have

$$\|P_t\|_{L^2(\mu)\to L^\infty(\mu)} \leqslant ce^{C_t}\exp\bigl[c_0\bigl(1+t^{-q/(q-2)}\bigr)\bigr]<\infty, \quad t>0.$$

In particular, P_t is uniformly integrable in $L^2(\mu)$.

Since (2.16) also implies P_t has a density w.r.t. μ, the compactness of P_t follows from Lemma 3.1 in [12]. □

3. Applications of main results

We first prove a general inequality in Hilbert space, which implies the dissipativity of the p-Laplace operator.

Lemma 3.1. *Let $(H,\langle\cdot,\cdot\rangle,\|\cdot\|)$ be a Hilbert space, then for any $0<r\leqslant 1$ we have*

$$\bigl\langle \|a\|^{r-1}a-\|b\|^{r-1}b, a-b\bigr\rangle \geqslant r\|a-b\|^2\bigl(\|a\|\vee\|b\|\bigr)^{r-1}, \quad a,b\in H. \tag{3.1}$$

Proof. Without loss of generality we may assume $\|a\|\geqslant\|b\|$, then (3.1) is equivalent to

$$\bigl(\|b\|^{r-1}-\|a\|^{r-1}\bigr)\langle b,a-b\rangle \leqslant (1-r)\|a\|^{r-1}\|a-b\|^2.$$

By the Cauchy–Schwarz inequality and the Young inequality we have

$$\begin{aligned}\bigl(\|b\|^{r-1}-\|a\|^{r-1}\bigr)\langle b,a-b\rangle &\leqslant \bigl(\|b\|^{r-1}-\|a\|^{r-1}\bigr)\|b\|\|a-b\|\\ &= \bigl(\|b\|^r\|a\|^{1-r}-\|b\|\bigr)\|a\|^{r-1}\|a-b\|\\ &\leqslant \bigl(r\|b\|+(1-r)\|a\|-\|b\|\bigr)\|a\|^{r-1}\|a-b\|\\ &\leqslant (1-r)\|a\|^{r-1}\|a-b\|^2.\end{aligned}$$

Hence the proof is complete. □

For the application of the main results, one need to verify the assumption (1.7). Here we give one sufficient condition such that (1.7) holds.

Proposition 3.2. *Suppose $Be_i=b_ie_i$ for $i\geqslant 1$, where $\{e_i\}$ is one orthonormal basis on $L^2(\Lambda)$. If there exists a constant $\sigma\geqslant 2$ such that*

$$B^{-\frac{\sigma}{2}}: H_0^{1,p}(\Lambda)\to L^2(\Lambda)$$

is a bounded operator, then (1.7) holds for the same exponent σ.

Proof. By assumption there exists a constant $C>0$ such that

$$\bigl\|B^{-\frac{\sigma}{2}}x\bigr\|_2^2=\sum_{i=1}^\infty\langle x,e_i\rangle^2 b_i^{-\sigma}\leqslant C\|x\|_{1,p}^2, \quad \forall x\in H_0^{1,p}(\Lambda).$$

Then by the Hölder inequality we have

$$\begin{aligned}\|x\|_B^\sigma &= \Biggl(\sum_{i=1}^\infty\langle x,e_i\rangle^2 b_i^{-2}\Biggr)^{\sigma/2}\\ &= \Biggl(\sum_{i=1}^\infty\langle x,e_i\rangle^{\frac{2\sigma-4}{\sigma}}\langle x,e_i\rangle^{\frac{4}{\sigma}}b_i^{-2}\Biggr)^{\sigma/2}\\ &\leqslant \Biggl(\sum_{i=1}^\infty\langle x,e_i\rangle^2\Biggr)^{\frac{\sigma-2}{2}}\Biggl(\sum_{i=1}^\infty\langle x,e_i\rangle^2 b_i^{-\sigma}\Biggr)\end{aligned}$$

$$= \|x\|_2^{\sigma-2}\left(\sum_{i=1}^{\infty}\langle x, e_i\rangle^2 b_i^{-\sigma}\right)$$
$$\leqslant C\|x\|_2^{\sigma-2}\|x\|_{1,p}^2. \tag{3.2}$$

Hence (1.7) holds. □

Corollary 3.3. *If Λ is a bounded C^∞-domain in $\mathbb{R}^d$ and $B = (-\Delta)^{-\theta}$ with $\theta \in (\frac{d}{4}, \frac{(2+d)p-2d}{8}]$, then B is a Hilbert–Schmidt operator and* (1.7) *holds for $\sigma = \frac{4}{p}$.*

Proof. It is well known that there exists an ONB $\{e_i\}$ on $L^2(\Lambda)$ such that

$$\Delta e_i = -\lambda_i e_i, \quad i \geqslant 1,$$

where the corresponding eigenvalues satisfy

$$\lambda_i \geqslant c \cdot i^{2/d}, \quad i \geqslant 1$$

for some constant $c > 0$. Hence for $\theta > \frac{d}{4}$ we have

$$\|B\|_{\mathrm{HS}}^2 = \sum_{i=1}^{\infty}\|Be_i\|_2^2 = \sum_{i=1}^{\infty}(\lambda_i)^{-2\theta} \leqslant C\sum_{i=1}^{\infty} i^{-4\theta/d} < \infty,$$

i.e. B is a Hilbert–Schmidt operator on $L^2(\Lambda)$.

By Proposition 3.2 it is enough to show $(-\Delta)^{\frac{\sigma\theta}{2}}$ is a bounded operator from $W_0^{1,p}(\Lambda)$ to $L^2(\Lambda)$.

Note that $(-\Delta)^{\frac{\sigma\theta}{2}}$ is a bounded operator from $H^{\sigma\theta,2}(\Lambda)$ to $L^2(\Lambda)$, where $H^{\sigma\theta,2}(\Lambda)$ is a fractional Sobolev space (cf. [31]).

By the general embedding theorem [31, Theorem 1, p. 82] we have for $\theta \leqslant \frac{(2+d)p-2d}{2p\sigma}$ the following embedding

$$W_0^{1,p}(\Lambda) \subseteq H^{\sigma\theta,2}(\Lambda)$$

is continuous, hence $(-\Delta)^{\frac{\sigma\theta}{2}}$ is a bounded operator from $W_0^{1,p}(\Lambda)$ to $L^2(\Lambda)$. □

Remark 3.1. For $d = 1$ we can take $B = (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{3p-2}{8}]$, where Δ is the Dirichlet Laplace operator on a bounded interval in $\mathbb{R}$. In this case the main results can only be applied to the case $p > \frac{4}{3}$. In particular, if $p = 2$, then we can take $\sigma = 2$ and $B = (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{1}{2}]$.

Example 3.4 *(Stochastic reaction–diffusion equations).* Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary and Δ the Laplace operator on $L^2(\Lambda)$ with Dirichlet boundary condition. Consider the following triple

$$H_0^1(\Lambda) \cap L^q(\Lambda) \subseteq L^2(\Lambda) \subseteq \left(H_0^1(\Lambda) \cap L^q(\Lambda)\right)^*$$

and the stochastic reaction–diffusion equation

$$\mathrm{d}X_t = \left(\Delta X_t - |X_t|^{q-2}X_t\right)\mathrm{d}t + B\,\mathrm{d}W_t, \qquad X_0 = x \in L^2(\Lambda), \tag{3.3}$$

where $q \geqslant 2$, B and W_t are Hilbert–Schmidt operator and cylindrical Wiener process on $L^2(\Lambda)$ respectively, then all assertions in Theorem 1.1 hold.

Moreover, if B is a one-to-one operator such that

$$B^{-1}: H_0^1(\Lambda) \to L^2(\Lambda)$$

is a bounded operator, then (1.7) holds. Therefore, all assertions in Theorem 1.2 and 1.3 also hold for (3.3).

In particular, if $d = 1$ and $B := (-\Delta)^{-\theta}$ with $\theta \in (\frac{1}{4}, \frac{1}{2}]$, then the associate transition semigroup of (3.3) is hyperbounded. If $q > 2$, then the corresponding transition semigroup of (3.3) is ultrabounded and compact.

Remark 3.2. (i) If we assume that

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$$

are the eigenvalues of $-\Delta$, the corresponding eigenvectors $\{e_i\}_{i\geqslant 1}$ form an ONB of $L^2(\Lambda)$. Suppose $Be_i := b_i e_i$ and there exists a positive constant C such that

$$\sum_i b_i^2 < +\infty \quad \text{and} \quad b_i \geqslant \frac{C}{\sqrt{\lambda_i}}, \quad i \geqslant 1, \tag{3.4}$$

then B is a Hilbert–Schmidt operator and (1.7) holds.

On the other hand, by the Sobolev inequality (see [37, Corollaries 1.1 and 3.1]) we know that

$$\lambda_i \geqslant c i^{2/d}, \quad i \geqslant 1,$$

for some constant $c > 0$. Hence (3.4) implies that the space dimension $d < 2$. However, if we can consider a general negative definite self-adjoint operator L instead of Δ in (3.3), e.g. $L := -(-\Delta)^\alpha$, $\alpha > 0$. Then, by means of the spectral representation, we can apply our results to some examples on $\mathbb{R}^d$ for $d \geqslant 2$. For more details we refer to [19,39].

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References

[1] S. Aida, Uniformly positivity improving property, Sobolev inequalities and spectral gap, J. Funct. Anal. 158 (1998) 152–185.
[2] S. Aida, H. Kawabi, Short time asymptotics of certain infinite dimensional diffusion process, in: Stochastic Analysis and Related Topics, VII, Kusadasi, 1998, in: Progr. Probab., vol. 48, 2001, pp. 77–124.
[3] S. Aida, T. Zhang, On the small time asymptotics of diffusion processes on path groups, Potential Anal. 16 (2002) 67–78.
[4] M. Arnaudon, A. Thalmaier, F.-Y. Wang, Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below, Bull. Sci. Math. 130 (2006) 223–233.
[5] A. Bensoussan, R. Temam, Equations aux derives partielles stochastiques non linéaires, Israel J. Math. 11 (1972) 95–129.
[6] S.G. Bobkov, I. Gentil, M. Ledoux, Hypercontractivity of Hamilton–Jacobi equations, J. Math. Pures Appl. 80 (7) (2001) 669–696.
[7] G. Da Prato, A. Debussche, L. Tubaro, Coupling for some partial differential equations driven by white noise, Stochastic Process. Appl. 115 (2005) 1384–1407.
[8] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia Math. Appl., Cambridge University Press, 1992.
[9] E. DiBenedetto, Degenerate Parabolic Equations, Springer-Verlag, New York, 1993.
[10] J.L. Doob, Asymptotics properties of Markoff transition probabilities, Trans. Amer. Math. Soc. 63 (1948) 393–421.
[11] I. Gyöngy, On stochastic equations with respect to semimartingale III, Stochastics 7 (1982) 231–254.
[12] F.-Z. Gong, F.-Y. Wang, Heat kernel estimates with application to compactness of manifolds, Q. J. Math. 52 (2) (2001) 171–180.
[13] M. Harrier, Exponential mixing properties of stochastic PDEs through asymptotic coupling, Probab. Theory Related Fields 124 (2002) 345–380.
[14] H. Kawabi, The parabolic Harnack inequality for the time dependent Ginzburg–Landau type SPDE and its application, Potential Anal. 22 (2005) 61–84.
[15] N.V. Krylov, B.L. Rozovskii, Stochastic evolution equations, translated from Itogi Nauki i Tekhniki, Sovrem. Probl. Mat. 14 (1979) 71–146.
[16] S. Kuksin, A. Shirikyan, Ergodicity for the randomly forced 2D Navier–Stokes equations, Math. Phys. Anal. Geom. 4 (2001) 147–195.
[17] S. Kuksin, A. Shirikyan, Coupling approach to white-forced nonlinear PDEs, J. Math. Pures Appl. 81 (2002) 567–602.
[18] O.A. Ladyzenskaja, New equations for the description of the motions of viscous incompressible fluids, and global solvability for their boundary value problems, Proc. Steklov Inst. Math. 102 (1967) 95–118; English transl. in: Tr. Mat. Inst. Steklova 102 (1967) 85–104.
[19] W. Liu, F.-Y. Wang, Harnack inequality and strong Feller property for stochastic fast diffusion equations, J. Math. Anal. Appl. 342 (2008) 651–662.
[20] W. Liu, Large deviations for stochastic evolution equations with small multiplicative noise, Appl. Math. Optim., in press.
[21] C. Mueller, Coupling and invariant measures for the heat equation with noise, Ann. Probab. 21 (1993) 2189–2199.
[22] B. Maslowski, J. Seidler, Strong Feller infinite-dimensional diffusions, in: Proceedings, Trento, 2000, in: Lect. Notes Pure Appl. Math., Marcel Dekker, 2002, pp. 373–389.
[23] J.C. Mattingly, Ergodicity of 2D Navier–Stokes with random forcing and large viscosity, Comm. Math. Phys. 206 (1999) 273–288.
[24] J.C. Mattingly, Exponential convergence for the stochastically forced Navier–Stokes equations and other partially dissipative dynamics, Comm. Math. Phys. 230 (2002) 421–462.
[25] C. Odasso, Ergodicity for the stochastic complex Ginzburg–Landau equations, Ann. Inst. H. Poincaré Probab. Statist. 42 (4) (2006) 417–454.
[26] E. Pardoux, Equations aux dérivées partielles stochastiques non linéaires monotones, Thesis, Université Paris XI, 1975.
[27] C. Prévôt, M. Röckner, A Concise Course on Stochastic Partial Differential Equations, Lecture Notes in Math., vol. 1905, Springer-Verlag, 2007.
[28] J. Ren, M. Röckner, F.-Y. Wang, Stochastic generalized porous media and fast diffusion equations, J. Differential Equations 238 (2007) 118–152.
[29] M. Röckner, F.-Y. Wang, Supercontractivity and ultracontractivity for (non-symmetric) diffusion semigroups on manifolds, Forum Math. 15 (2003) 893–921.
[30] M. Röckner, F.-Y. Wang, Harnack and functional inequalities for generalized Mehler semigroups, J. Funct. Anal. 203 (2003) 237–261.
[31] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, de Gruyter Ser. Nonlinear Anal. Appl., vol. 3, Walter de Gruyter, Berlin, 1996.
[32] J. Seidler, Ergodic behaviour of stochastic parabolic equations, Czechoslovak Math. J. 47 (1997) 277–316.
[33] J.L. Vázquez, Smoothing and Decay Estimates for Nonlinear Diffusion Equations, Oxford Lecture Ser. Math. Appl., vol. 33, Oxford University Press, 2006.
[34] J.L. Vázquez, The Porous Medium Equation: Mathematical Theory, Oxford Math. Monogr., Oxford University Press, 2007.
[35] J.B. Walsh, An introduction to stochastic partial differential equations, in: P.L. Hennequin (Ed.), Ecole d'Ete de Probabilite de Saint-Flour XIV, in: Lecture Notes in Math., vol. 1180, 1984, pp. 265–439.
[36] F.-Y. Wang, Logarithmic Sobolev inequalities on noncompact Riemannian manifolds, Probab. Theory Related Fields 109 (1997) 417–424.
[37] F.-Y. Wang, Functional inequalities, semigroup properties and spectrum estimates, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 3 (2000) 263–295.
[38] F.-Y. Wang, Equivalence of dimension-free Harnack inequality and curvature condition, Integral Equations Operator Theory 48 (2004) 547–552.
[39] F.-Y. Wang, Harnack inequality and applications for stochastic generalized porous media equations, Ann. Probab. 35 (2007) 1333–1350.
[40] X. Zhang, On stochastic evolution equations with non-Lipschitz coefficients, BiBoS-Preprint 08-03-279.